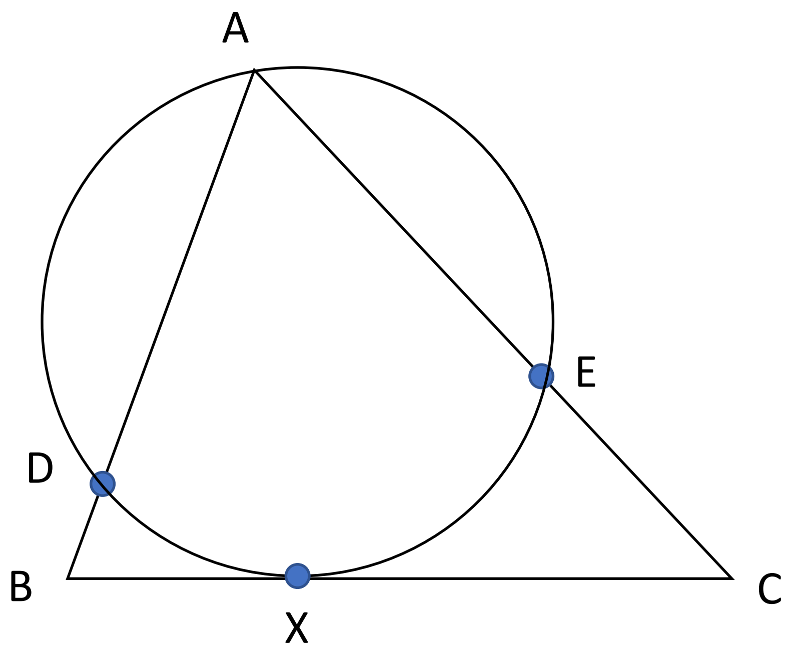
Problem Solving 4

Lecture 15 Apr 18, 2021

- Q1 (Leftover from last week). Suppose the triangle ABC has acute angles. Let D and E be points on AB and AC such that the circumcircle of the triangles ADE is tangent to BC at the point X. If we choose D and E so that the distance DE is minimum, then which of the following statements is correct:

- [ ] X is the midpoint of BC
- [ ] AX is perpendicular to BC
- [ ] AX is the angle bisector of A
- [ ] None



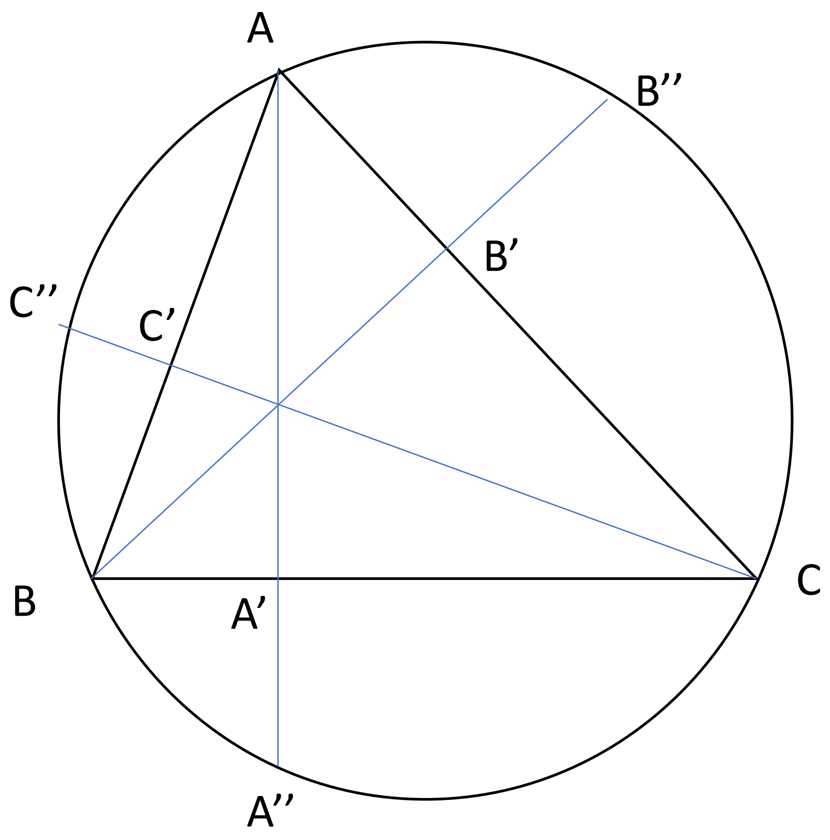
- Q2 (leftover from last week)
- We have 2021 lamps $L_1, \ldots, L_{2021}$ that are all off at the beginning.
- We also have 2021 switches $P_1, \ldots, P_{2021}$.
- Pressing the switch P_k will turn On/Off all the lamps L_n where n is a multiple of k. (For example, L_2 will turn On/Off all even numbered lamps).

- If we press all of $P_1, \ldots, P_{2021}$ once in order, at the end, how many lamps will be ON?

- Q3. We chosen a subset S of $\{1,2,\ldots,1001\}$ so that if $x, y \in S$ then $x + y \notin S$. What is the maximum possible size of S.

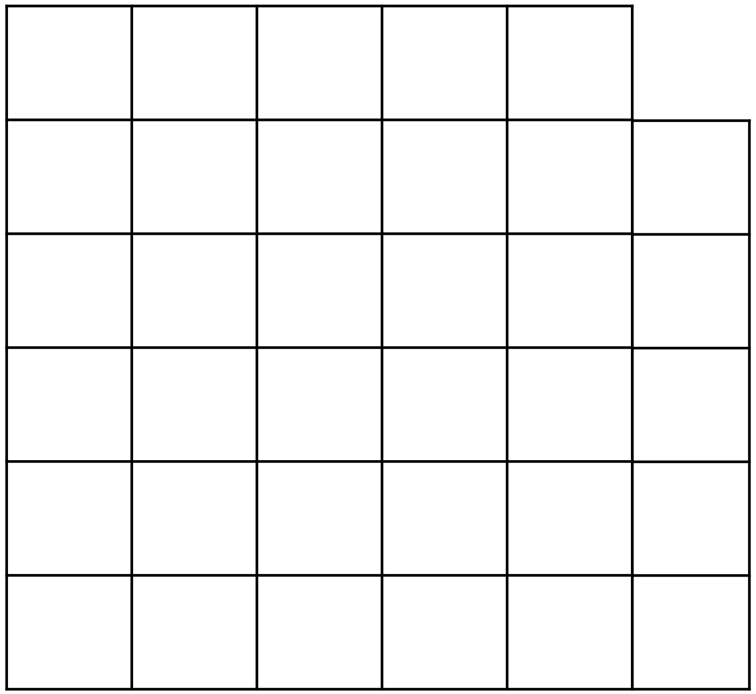
- Q4. Let AA', BB', and CC' be the the 3 heights of the triangle ABC. We continue these until they intersect the circumscribed circle of ABC at the points A", B", C". If AB=4, AC=5, BC=6 then find

$$\frac{AA''}{AA'} + \frac{BB''}{BB'} + \frac{CC''}{CC'}$$



- Q5. Suppose $n \geq 2$ and $n^2 + 2^n$ is a prime number. What are the possible value of n modulo 6.

- Q6. Count the number of squares in the following picture.



- Q7. The sequence

$$a_0, a_1, a_2, \ldots$$

is defined by

$$a_0 = 0 \quad a_{n+1} = na_n + a_n + n$$

What is the value of $a_{101} \ modulo \ 102$?

- Q8. Find the solutions (x, y) of the equation

$$2x^2y^2 + y^2 = 26\,x^2 + 1201$$

in natural numbers.